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A Critique of the Disturbance Theory of Indeterminacy in Quantum Mechanics[1]

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Heisenberg's gedanken *experiments in quantum mechanics have given rise to a widespread belief that the indeterminacy relations holding for the variables of a quantal system can be explained quasiclassically in terms of a disturbance suffered by the system in interaction with a quantal measurement, or state preparation, agent. There are a number of criticisms of this doctrine in the literature, which are critically examined in this article and found to be in-inconclusive, the chief error being the conflation of this disturbance with the projection postulate. We present a critique of the disturbance theory based on the fact that the required disturbance will in general depend on the interaction time of the system and state-preparer. This point is exploited in the construction of a spin-interaction model which acts as a counterexample to the disturbance doctrine, while remaining faithful to the spirit of Heisenberg's* gedanken *experiments. Several consequences of this result are discussed.*

1. INTRODUCTION

In the foundations of elementary quantum mechanics (QM) the well-known problems concerning the theoretical treatment of the measuring process may be conveniently separated into two compartments. In the first, there are considerations pertaining to the effect that such a process has on the measuring instrument. How can the final macroscopic states of the pointer on the instrument be reconciled with the linear laws of QM? Such a question

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has received enormous attention in the literature and is perhaps the most fundamental issue in the foundations of the theory. On the other hand, there are questions concerned with the effect of the observation interaction on the object system. In this respect, a prevailing assumption is, loosely speaking, that the object system is somehow irreducibly and unpredictably disturbed in the course of observation, and moreover that this disturbance is the seat of the indeterminacy which is a characteristic feature of each quantal system.

It is with this second issue that we are concerned in this paper. We shall attempt to clarify the content of the disturbance theory, and after an analysis of several existing critical discussions of it, which we find to be severally inconclusive, we construct a counterexample to the theory.

The disturbance doctrine is an offspring of the celebrated analysis due to Heisenberg in the late twenties of idealized measurement processes.[4] Heisenberg in fact uses two sorts of argument in the course of his discussion.

In the case of the microscope experiment, and also the Doppler momentum-measurement experiment, an incoming light particle (which may be considered the measuring system) collides with an electron (the object system), the collision being governed by the classical laws of conservation of energy and linear momentum. In both cases, the light particle is endowed with quantal fluctuations which correspond to the use of the Einstein–de Broglie relations for the energy and momentum for these systems. The electron may be effectively considered a classical particle, which in interaction with the quantal measuring agent gains quantal fluctuations, as witnessed by the final indeterminacy relations obtained for it.[5]

However, in discussing the diffraction of an electron beam by a slit, Heisenberg derives the indeterminacy relations for the electrons emerging from the slit using quantal aspects of the electrons, not of the diaphragm containing the slit (Ref. 1, pp. 23–24). In subsequent discussion of the single slit experiment by Bohr (Ref. 4, pp. 214–15), however, quantal aspects of the diaphragm containing the slit were regarded as responsible for the momentum uncertainty in the emerging beam. So Heisenberg's second sort of example is apparently reducible to the first. We shall argue in what follows that this additional analysis supplied by Bohr is essentially misleading as an

[4] For details of the *gedanken* experiments, see Heisenberg.[(1)] For a historical account of their development, see Jammer.[(2)]

[5] Recently, Roychouldhuri[(3)] has remarked in relation to the microscope experiment that later improvements in resolution techniques render the Rayleigh criterion of the finite resolving power of the microscope overly restrictive. It was the Rayleigh criterion that Heisenberg, following a suggestion by Bohr, used in deriving the indeterminacy relations. In such cases of "super resolution," in principle Heisenberg's general arguments may be used to violate the indeterminacy relations.

explanation of quantal indeterminacy for the electron in the single-slit experiment, and that Heisenberg's original analysis in terms of quantal aspects of the electron is the line that should be uniformly adopted in discussing the other experiments, such as the Doppler momentum measurement, etc.

Indeed Bohr himself in his later writings retreated from the disturbance doctrine as an explanation of the uncertainty relations, emphasizing the wholeness of a quantal phenomenon involving the specification of the experimental arrangement in classical terms, rather than mechanical transmission of uncontrollable disturbance as the source of the characteristic features of the theory. This shift was apparently due to difficulties of understanding the Einstein–Podolsky–Rosen discussion in terms of a disturbance theory.[6] We shall comment on this in Section 2.

A version of the disturbance theory can also be found in the classic Bohr and Rosenfeld[(7)] paper on the quantization of the radiation field. In this work, the authors developed a quasiclassical analysis of field measurement, where the field (the object system) and measurement process are treated classically, except that the material test charge (the measuring agent) is subject to quantal indeterminacies. This treatment yielded indeterminacies in the measured field strengths which exactly match those yielded by the formal quantum theory of radiation.

In all these discussion, then, the originally well-defined observables corresponding to the respective object systems are considered to be "disturbed" in the course of the interaction, and the resulting indeterminacies in the values of the observables are consistent with the familiar theoretical dispersion relations for self-adjoint operators on Hilbert space. In a famous passage Heisenberg expressed the situation as follows: "...the interaction between observer and object causes uncontrollable and large changes in the system being observed because of the discontinuous changes characteristic of atomic processes."[(1)]

In more recent times, a certain degree of clarity has been gained in the discussion of such questions by distinguishing between measurement processes and state-preparation processes. Strictly speaking, insofar as they are con-concerned with post-interaction ensembles for the object system, the *gedanken* experiments we are referring to are concerned with state preparations and not measurements. Heisenberg[(1)] himself stressed that the uncertainty relations "do not refer to the past,"[7] i.e., to the preinteraction ensemble. This point is important when the indeterminacy relations derived in the

[6] Bohr,[(5)] p. 700. Also see Schiebe,[(6)] pp. 20 and 26.

[7] Heisenberg[(1)] was not always consistent in this respect, however. In the Doppler momentum-measurement experiment, $\Delta p_y \, \Delta y \sim h$ is derived for the precollision electron. But the result goes through, from the same premisses, for the postcollision electron as well.

gedanken experiments are compared with those derived directly from quantum theory. If A is a self-adjoint operator on Hilbert space, the standard deviation ΔA is defined as follows:

$$\Delta A = [\langle A^2 \rangle - \langle A \rangle^2]^{1/2} \tag{1}$$

where $\langle \cdot \rangle$ denotes mean value. It follows from the properties of Hilbert space that for any self-adjoint operators A and B

$$\Delta A \cdot \Delta B \geqslant \tfrac{1}{2} \, |\langle [A, B] \rangle| \tag{2}$$

where [,] denotes the commutator bracket.[8] The standard deviations ΔA and ΔB correspond to the dispersion of A and B measurement outcomes[9] predicted over an ensemble of similarly prepared systems in a pure state. The relation (2) asserts the impossibility of preparing a dispersion-free quantal ensemble.[10]

We can now express the content of the disturbance theory more precisely as follows. The relation (2) is the formal conterpart of indeterminacy in the observables of the object system caused by a disturbance in the state-preparation procedure, which produces uncontrollable fluctuations in the incompatible variables. The general thesis that quantal indeterminacy in a system I is a result of physical interaction with a quantal state-preparing agency II, we refer to henceforth as the disturbance theory of indeterminacy (DTI).

2. EXISTING CRITICISMS

There is a tendency to regard the projection postulate within the formal theory of measurement in QM as the formal expression of the disturbance phenomenon (e.g., Ref. 14, pp. 197–98; Ref. 15, p. 418). Indeed, the unpredictable, discontinuous nature of the ψ-state evolution for the object system as entailed by the postulate establishes a certain parallelism with the DTI. Taking the simplest case of measurement of a nondegenerate observable

[8] Robertson.[(8)] For a more general expression for $\Delta A \cdot \Delta B$, see Schrödinger[(9)] or Synge.[(10)] Notice that if for a certain pair A and B it happens that $[A, B]$ is not a constant of the motion, then the minimum value for $\Delta A \cdot \Delta B$ will be a time-dependent quantity. Such is the case with angular momentum operators, as we shall see in Section 3.

[9] We denote by A and B both the operators and the corresponding observables.

[10] That the indeterminacy relations are concerned with state-preparation procedures and not measurements, and that they place no constraints on the accuracy of individual measurements, are points that have been stressed, albeit controversially, by a growing number of commentators. See, for example, Margenau,[(11)], Popper,[(12)] and Ballentine.[(13)]

O with a purely discrete spectrum,[11] the projection postulate can be formulate as follows in the usual ensemble interpretation of the statistical algorithm of quantum mechanics. If the density operator for the initial ensemble represented by a pure state ψ is $\rho = P_\psi$, then the density operator appropriate to the subensemble obtained by selecting those systems that yielded a value a_n on measuring the observable O is $\rho'' = P_{\alpha_n}$, where α_n is the eigenvector corresponding to the measured value a_n. The transition $\rho \to \rho''$ is made up of two distinct steps. First we have the transition $\rho \to \rho' = \sum_n |\langle\alpha_n \mid \psi\rangle|^2 P_{\alpha_n}$ which is induced in the original ensemble by the *physical* interaction with the measurement apparatus. This change in density operator arises from what may be termed nonselective measurement. The additional change $\rho' \to \rho''$ arises from the *selection* of the appropriate subensemble from the post-measurement ensemble described by ρ'. This latter transition is not of course produced by any physical effect.

Various questions arise. If the measurement constitutes a state preparation for an ensemble of systems described by the state vector α_n, then the satisfaction of the projection postulate is *necessarily* true since the trantransition to any density operator other than P_{α_n} would by definition not constitute the indicated state preparation.[12] Having settled the question of whether the projection postulate is true with respect to state preparation, we can ask whether the transition $\rho \to \rho'$ can in any way be subsumed under causal Schrödinger-type time evolution. This is the central problem of the so-called theory of measurement in QM, which is outside the scope of this paper. Next, what is the relation between the transition $\rho \to \rho''$, or more simply $\psi \to \alpha_n$, which expresses the projection postulate, and the DTI? Suppose ψ is already an eigenstate of some operator O' which does not commute with O. Then in the final state α_n, O' will become indeterminate. This is supposed to be explained by the DTI insofar as it affords an explanation of the uncertainty relations appropriate to the state α_n. But the DTI does not explain the transition $\psi \to \alpha_n$, which shows how one quantum mechanical state is replaced by another; rather, its purported function is to explain quasiclassically the origin of the indeterminacy relations appoicable to both the states ψ and α_n with respect to the noncommuting observables O and O' in terms of the state-preparation procedures for the initial state ψ or the final state α_n.

Conflation of the disturbance of the ψ state described in the projection postulate with the disturbance described in the informal DTI is prevalent in some discussions of proposed objections to the DTI. Thus Park[(16)] has

[11] For a discussion of how to formulate the projection postulate for measurement of degenerate observables see Herbut,[(33)] who gives references to the relevant literature.

[12] For measurements which do not constitute state preparations of the sort described, the projection postulate is of course *necessarily* false.

constructed a formal measurement scheme in which two distinguishable spin-$\frac{1}{2}$ systems (one playing the role of the object, the other the instrument) interact in such a way that the ψ state of the object system is not disturbed, but is "transferred" to the instrument. Park concludes that disturbance is not an inevitable feature of the measurement act. But the argument is irrelevant to the DTI since Park's measurement scheme is not a state-preparation device, and the disturbance which Park eliminates in his example is anyway referring to the alteration in the quantum mechanical state, which as we have seen is a different animal from that envisaged in the DTI. Indeed the DTI is logically prior to any formal theory of measurement in QM since it presupposes a quantal nature for the instrument (state-preparer) only, and it makes no sense in the DTI to suppose that the ψ state of the object must undergo continuous, or for that matter, discontinuous (as when the projection postulate is presupposed) motion in measurement.

Another objection to the DTI concerns the behavior of correlated, distant systems of the kind first discussed so significantly by Einstein *et al.*[17] Let us consider, in the manner of Bohm and Aharonov,[18] a pair of separated spin-$\frac{1}{2}$ systems A, B governed at time t by the spherically symmetric wave function

$$\Psi(t) = (1/\sqrt{2})[\psi_+(A) \otimes \psi_-(B) - \psi_-(A) \otimes \psi_+(B)]$$

where $\psi_-(A)$ is the spin-up state of A, etc.

Now suppose σ_x is chosen for measurement on A at t, yielding the result $+1$. Then in accordance with the projection postulate, system B is immediately after the measurement in the state $\psi_-(B)$, which means that the value of σ_x on B is -1, and those of σ_y and σ_z on B are indeterminate. Bohm and Aharonov argue that this result is incompatible with the DTI, since the disturbance in the measurement on A first "does not explain why particle B ... realizes its potentiality for a definite spin in precisely the same direction as that of A" and second "it cannot explain the fluctuations of the other two components of the spin of particle B."[13]

Let us take these two points in order. As to the first, the DTI does not, it is true, explain how the value of σ_x on B becomes definite in response to the distant measurement on A. Of course, this result is a direct outcome of the correlations built into the state $\Psi(t)$, and if a further (local) explanation

[13] Ref. 18. It may be noted that these authors do not conclude that the disturbance theory is false, but on the contrary they assume its validity and so infer that the correlated behavior of A, B is paradoxical. This is quite a different interpretation of the "paradox" of such correlated systems from that found in the original paper of Einstein *et al.* In the above, we construe the argument of Bohm and Aharonov as an objection to the DTI, treating the correlation behavior as a straightforward theorem of QM.

is felt necessary, the DTI does not pretend to encompass such behavior. As we have defined it, the DTI is concerned with explaining the origin of quantal indeterminacy, and not with the manner in which the indeterminacies are resolved into definite values as a result of measurement, either local or otherwise. (Again, the misunderstanding may be expressed as arising in the conflation of disturbance in the DTI with the projection postulate.) As to the second point, the physical effect of measurement is not to create indeterminacies in the values of σ_y and σ_z on B, but rather to induce the nonselective transition

$$\rho = P_{\Psi(t)} \rightarrow \rho' = \tfrac{1}{2}[P_{\psi_+(A)\otimes\psi_-(B)} + P_{\psi_+(A)\otimes\psi_-(B)}]$$

The values of σ_x, σ_y, and σ_z were already indeterminate for B prior to the measurement on A according to the state $\Psi(t)$ and it is well known that the probabilistic properties of any spin component of B (excluding of course correlations with the spin components of A) are unchanged when ρ is replaced by ρ'. It would appear, in conclusion, that the features of distant correlated systems in QM do not furnish a convincing objection to the DTI.[14]

The final objection to the theory we shall consider is based on the phenomenon of potential barrier penetration in QM. If a particle with (expected) energy E enters a potential barrier of magnitude V, where $E < V$, then it would seem that the conservation of energy implies that in the region of potential V, the kinetic energy of the particle must be negative, which is impossible. The question then arises whether the missing energy of the particle in the classically inaccessible region is supplied by the disturbance produced in the measurement or by the state preparation of the particle prior to entry of the barrier. This latter possibility is ruled out by the following considerations[(21–23)]:

[14] Another argument, very similar to that of Bohm and Aharonov, which is based on the negative-result-type experiment of Renninger,[(19)] may be formulated against the DTI. In the celebrated two-slit experiment, if a detector placed at one of the two open slits fails to register the passage of the particle, this result is tantamount to the detection of the particle through the remaining slit. A specially chosen ensemble of such negative results would not yield the interference distribution at the detection screen, and thus the mere presence of the detector at the original slit is said to disturb the system without the detector entering into physical interaction with it. (This is essentially the principle underlying the recent argument against disturbance as a physical interaction of Yoshihuku,[(20)] although the author uses the more complicated example of a double Stern–Gerlach experiment. For related discussions, see the account in Jammer,[(2)] pp. 495–496.) However, as in the case of the correlated systems, this "disturbance" does not create indeterminacies in the manner which is of interest to us, and is similarly the physical expression of the projection postulate, here identified with the collapse of the wave function at the unguarded slit.

(a) The determination of the energy of the particle prior to penetration may be as gentle as we please.

(b) The energy gap $V–E$ can always be made larger than the interaction energy of the state preparation procedure.

We remark that both Bohm[(24)] and Fong[(25)] attempt to explain barrier penetration in terms of the disturbance suffered by the particle due to its detection once within the barrier. But this approach fails to explain quasi-classically how the particle made the penetration in the first place.

The fact that the DTI is incapable of elucidating the penetration process is considered by some to diminish its plausibility.

However, the motivation for introducing the DTI into the discussion, viz. the problem of negative kinetic energy, is misguided. We have for the Hamiltonian of the particle at position x

$$H(x) = p^2/2m + V(x)$$

The claim that the kinetic energy is negative in the potential barrier is a consequence of the belief that the total value of the energy of the system is the sum of the values of the kinetic and potential energy terms. However, the quantum mechanical law of conservation of energy states that $\langle H \rangle$ is constant, and is silent concerning the relationship between values of the kinetic and potential energies for individual particles.[15] Thus it does not follow that the kinetic energy of the particle in the classically inaccessible region is negative.[16]

3. A COUNTEREXAMPLE

In this section we examine, by way of a specific model, the relationship between the disturbance phenomenon in state preparation and the time of interaction between object and state-preparer. Our analysis is motivated by the simple observation that in state preparation processes that are not instantaneous, the swapping of indeterminacy from the quantal preparing agent to the object system in the quasiclassical description will generally be a function of the interaction period. Whether this should pose a difficulty to the DTI depends on the interaction in question.

In order to clarify this last point, let us consider a simple one-dimensional elastic collision (along the x axis) between identical particles of mass μ. The incoming particle 2 has initial momentum p_2. The particle 1 is assumed

[15] Compare Gardner[(26)] for a discussion of this point.

[16] All that we can infer is that $\langle V(x) \rangle < \langle H(x) \rangle = E$, so that the particle can never be confined *entirely* in the classically inaccessible region.

to be initially at rest in the laboratory frame; its postcollision momentum is p_1'.[17] Conservation of linear momentum and of energy implies

$$p_1' = p_2 , \qquad p_2' = 0$$

so that

$$\Delta p_1' = \Delta p_2$$

We assume that for particle 2 (the quantal agant)

$$\Delta p_2 \, \Delta x_2 \sim \hbar$$

As a result of the initial indeterminacy Δx_2, the instant of collision is indeterminate to the extent

$$T = \mu \, \Delta x_2 / p_2 \sim \mu \hbar / (p_2 \, \Delta p_1')$$

Thus the coordinate x_1 has the indeterminacy

$$\Delta x_1' = \frac{T p_1'}{\mu} \sim \frac{\mu \hbar}{p_2 \, \Delta p_1'} \frac{p_1'}{\mu} = \frac{\hbar}{\Delta p_1'}$$

So

$$\Delta p_1' \, \Delta x_1' \sim \hbar \tag{3}$$

As in Heisenberg's *gedanken* experiments, the "disturbance" suffered by the object system 1 results in the system acquiring quantal indeterminacy. Now we may imagine particles having force fields with finite ranges, with the result that they repel each other within some finite distance. Thus we can picture, for example, a finite Hooke's law spring, with mass negligable in relation to μ, lying along the x axis between the two particles prior to collision. In this case it can be shown that relation (3) obtains only after all the potential energy gained in the spring by the impingement of particle 2 is completely converted into the kinetic energy of particle 1, i.e., after particle 1 has decoupled itself from the spring.

In this case, the time taken for the spring to compress and relax marks the complete period of interaction for the "state preparation" of particle 1. However, in the next example, which will be developed in more detail, there is no similar naturally specified complete interaction period. In this way, it is ambiguous when the full swapping of indeterminacies should be expected to have occurred. The analysis of coupled spin systems suggests itself because it is capable of being straightforwardly treated both fully quantum mechanically, and quasiclassically in the present sense. This allows a detailed comparison to be made between the two treatments. We begin with the quantal treatment.

[17] The prime will be used to denote postcollision value.

3.1. Quantum Mechanical Treatment

Consider two spin-$\frac{1}{2}$ systems 1 and 2 which interact according to the interaction Hamiltonian

$$H_{\text{INT}} = g\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

where g is the intensity factor.[18] $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are assumed to commute with the free Hamiltonians for the individual systems. The total Hamiltonian is thus effectively

$$H = H_1 \otimes 1 + 1 \otimes H_2 + H_{\text{INT}} \sim g\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

Now, since

$$i\hbar\dot{\boldsymbol{\sigma}}_i = [\boldsymbol{\sigma}_i, H] \qquad \text{for} \quad i = 1, 2$$

then using the following relation for angular momentum operators in QM, where $\mathbf{A}$ is any vector operator commuting with $\boldsymbol{\sigma}$,

$$[\boldsymbol{\sigma} \cdot \mathbf{A}, \boldsymbol{\sigma}] = 2i\boldsymbol{\sigma} \times \mathbf{A}$$

we obtain

$$\begin{aligned} \dot{\boldsymbol{\sigma}}_1 &= -(2g/\hbar)(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \\ \dot{\boldsymbol{\sigma}}_2 &= (2g/\hbar)(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \\ \dot{\boldsymbol{\sigma}} &= 0 \qquad \text{where} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 \end{aligned} \tag{4}$$

We see that the total angular momentum is conserved. Suppose now that the initial ($t = 0$) state of the joint system is represented by

$$\Psi_0 = (1/\sqrt{2})\, \alpha(1) \otimes \{\alpha(2) + \beta(2)\}$$

where $\alpha(1)$ and $\beta(1)$ denote the spin-up and spin-down states, respectively, of system 1 in the z direction, etc. Ψ_0 most closely resembles the situation envisaged in the *gedanken* experiments where the object (system 1) is originally supposed to be in a well-defined state with respect to the observable being measured, here σ_z, and where the quantal agent (system 2) possesses indeterminacy in this respect.

The state of the joint system at $t > 0$ is given by

$$\Psi_t = \{\exp[-(i/\hbar)\, g(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\, t]\}\, \Psi_0$$

Now,

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = \tfrac{1}{2}(\sigma^2 - \sigma_1{}^2 - \sigma_2{}^2)$$

so that

$$\Psi_t = \{\exp[-(i/\hbar)\, g(\tfrac{1}{2}\sigma^2 - 3)\, t]\}\, \Psi_0 \tag{5}$$

[18] H_{INT} is defined on the tensor product $\mathscr{H}^2 \otimes \mathscr{H}^2$ of the two-dimensional Hilbert spaces representing systems 1 and 2. For simplicity we write $\boldsymbol{\sigma}_1$ instead of $\boldsymbol{\sigma} \otimes 1$, etc., where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are Pauli spin vector operators for the systems 1 and 2, respectively.

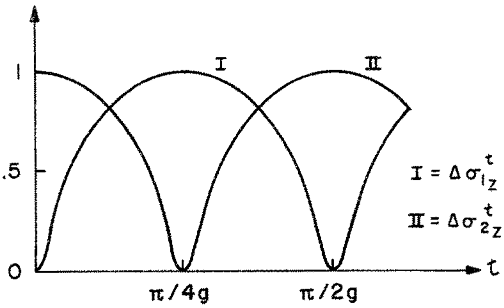

Fig. 1. Spin indeterminacy of systems 1, 2 plotted against time.

Now if $\{|(\sigma^2)_i\rangle\}$ is the complete set of eigenvectors of σ^2 in $\mathscr{H}^2 \otimes \mathscr{H}^2$ corresponding to eigenvalues $(\sigma^2)_i$, we can expand Ψ_0 in terms of this basis:

$$\Psi_0 = \sum_{i=1}^{4} a_i\, |(\sigma^2)_i\rangle, \qquad \text{where} \quad \sum_{i=1}^{4} |\, a_i\,|^2 = 1$$

From (5) we now have

$$\Psi_t = \left\{\exp\left[-\frac{i}{\hbar} g \left(\frac{1}{2}\sigma^2 - 3\right) t\right]\right\} \sum_i a_i\, |(\sigma^2)_i\rangle$$

$$= \sum_i a_i \left(\exp\left\{-\frac{ig}{\hbar}\left[\frac{1}{2}(\sigma^2)_i - 3\right] t\right\}\right) |(\sigma^2)_i\rangle$$

Expressed in terms of the α's and β's, the solution is[19] (putting $\hbar = 1$)

$$\Psi_t = (1/\sqrt{2})\{\alpha(1) \otimes \alpha(2) + \tfrac{1}{2}[1 + \exp(4igt)]\, \alpha(1) \otimes \beta(2)$$
$$+ \tfrac{1}{2}[1 - \exp(4igt)]\, \beta(1) \otimes \alpha(2)\} \tag{6}$$

Now we wish to derive the expressions for $\Delta\sigma_{1z}$ and $\Delta\sigma_{2z}$, the standard deviations of the z components of the spins of 1 and 2, respectively, as functions of time. From (1) and (6) we obtain

$$\Delta\sigma^t_{1(2)z} = \tfrac{1}{2}[\tfrac{5}{2} - (+)\, 2\cos 4gt - \tfrac{1}{2}\cos 8gt]^{1/2}$$

These expressions are plotted in Fig. 1. It is seen that the indeterminacy in the z component of the spin in system 2 is wholly transmitted to system 1 after an interaction period $t = \pi/4g$, and as the interaction continues, the process is reversed, and so on. Now the commutator brackets of angular

[19] For a matrix treatment of coupled spin-$\frac{1}{2}$ systems with the same Hamiltonian, see Park.[16]

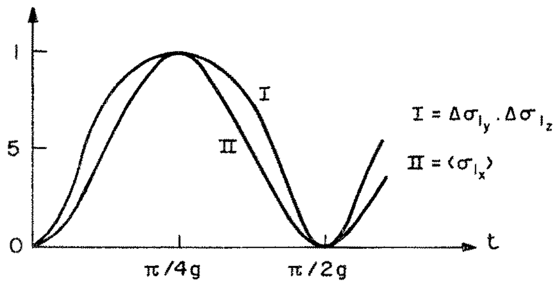

Fig. 2. The quantities $\Delta\sigma_{1y} \cdot \Delta\sigma_{1z}$ and $\langle\sigma_{1x}\rangle$ plotted against time.

momentum operators are not constants of the motion, giving rise to time-dependent indeterminacy relations (see Section 1). In particular, from (2) we have

$$\Delta\sigma_{1y}^{t} \cdot \Delta\sigma_{1z}^{t} \geqslant \langle\sigma_{1x}\rangle^{t} \tag{7}$$

where

$$\langle\sigma_{1x}\rangle^{t} = \tfrac{1}{2} - \tfrac{1}{2}\cos 4gt$$

In Fig. 2 we plot the values of $\Delta\sigma_{1y} \cdot \Delta\sigma_{1z}$ and $\langle\sigma_{1x}\rangle$ against time. Notice the periodic vanishing of the product $\Delta\sigma_{1y} \cdot \Delta\sigma_{1z}$, which, despite the fact that σ_{1y} and σ_{1z} do not commute, occurs at those times when $\langle\sigma_{1x}\rangle$ vanishes, as is consistent with (7).

3.2. The Quasiclassical Treatment

Analogously to the quantal coupled spins case, we start with a pair of classical spinning systems 1 and 2 with respective angular momentum vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ whose interaction is governed by the Hamiltonian

$$H = k\mathbf{L}_1 \cdot \mathbf{L}_2$$

We have

$$\dot{\mathbf{L}}_{1(2)} = \{\mathbf{L}_{1(2)}, H\}$$

where { , } is the Poisson bracket. It is readily verified that [compare with (4)]

$$\begin{aligned} \dot{\mathbf{L}}_1 &= -k\mathbf{L}_1 \times \mathbf{L}_2 \\ \dot{\mathbf{L}}_2 &= k\mathbf{L}_1 \times \mathbf{L}_2 \\ \dot{\mathbf{L}} &= 0 \qquad \text{where} \quad \mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \end{aligned}$$

Now using

$$\mathbf{L}_1 \times \mathbf{L}_1 = \mathbf{L}_2 \times \mathbf{L}_2 = 0$$

we obtain

$$\dot{\mathbf{L}}_1 = -k\mathbf{L}_1 \times \mathbf{L}, \qquad \dot{\mathbf{L}}_2 = -k\mathbf{L}_2 \times \mathbf{L} \tag{8}$$

Equations (8) imply that the vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ precess about the vector $\mathbf{L}$ with a fixed angular velocity. If we take $|\mathbf{L}_1| = |\mathbf{L}_2|$, the tips of the vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ lie in the same plane perpendicular to $\mathbf{L}$. This behavior is depicted in Fig. 3, where the angular displacement θ is given by $\theta = kLt$, where $L = |\mathbf{L}|$.

Now in order to mimic the quasiclassical treatment of the *gedanken* experiments discussed in Section 1, we allow the initial ($t = 0$) $\mathbf{L}_2$ components to have "quantal" indeterminacy, denoted by ΔL_{2i}^0 ($i = x, y, z$). What is of interest is the way this indeterminacy in $\mathbf{L}_2$ is transmitted to system 1 (the object system) as it interacts with system 2 (the quantal agent).

In solving for $\mathbf{L}_1{}^t$, $\mathbf{L}_2{}^t$ for time $t > 0$, we shall make use of the tensor treatment of rotation (Ref. 27, p. 96). If $\mathbf{L}$ has the direction cosines n_i ($i = x, y, z$), and $\mathbf{L}_{1(2)}^0$ represents the angular momentum vector of system 1 (2) at $t = 0$, then the individual components of $\mathbf{L}_1{}^t$, say, are given in the summation convention by

$$L_{1i}^t = R_{ik} L_{1k}^0 \qquad (i = x, y, z) \tag{9}$$

with

$$R_{ik} = \cos\theta\, \delta_{ik} + (1 - \cos\theta)\, n_i n_k - \epsilon_{ikl} \sin\theta\, n_l \tag{10}$$

where ϵ_{ikl} is the Levi-Civita symbol. The same identity (9) holds for L_{2i}^t.

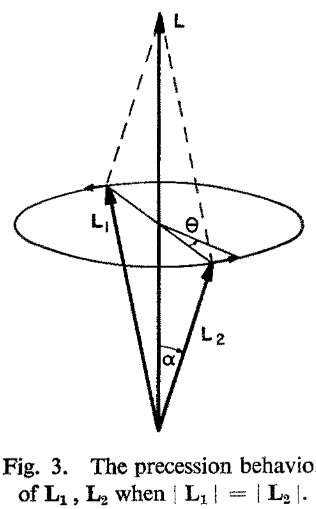

Fig. 3. The precession behavior of $\mathbf{L}_1$, $\mathbf{L}_2$ when $|\mathbf{L}_1| = |\mathbf{L}_2|$.

Now in solving for $\Delta L^t_{1(2)i}$ we shall go only to first order in ΔL^0_{2i}. This simplifies the calculations and it is a reasonable approximation if the original indeterminacies ΔL^0_{2i} are small in comparison with $|\mathbf{L}_1{}^0| = |\mathbf{L}_2{}^0|$.

We have from (9) and (10)

$$\begin{aligned} \Delta L^t_{1i} &= \frac{\partial}{\partial L^0_{2m}} (R_{ik} L^0_{1k})\, \Delta L^0_{2m} \\ \Delta L^t_{2i} &= \frac{\partial}{\partial L^0_{2m}} (R_{ik} L^0_{2k})\, \Delta L^0_{2m} \end{aligned} \tag{11}$$

Now writing

$$\Lambda_{mik} = \frac{\partial}{\partial L^0_{2m}} R_{ik} \tag{12}$$

we obtain from (11) by differentiation

$$\begin{aligned} \Delta L^t_{1i} &= \Lambda_{mik} L^0_{1k}\, \Delta L^0_{2m} \\ \Delta L^t_{2i} &= \Lambda_{mik} L^0_{2k}\, \Delta L^0_{2m} + R_{ik}\, \Delta L^0_{2k} \end{aligned} \tag{13}$$

Finally, using the identities

$$\frac{\partial \theta}{\partial L^0_{2m}} = \frac{\theta n_m}{L}, \qquad \frac{\partial n_i}{\partial L^0_{2m}} = \frac{1}{L} (\delta_{im} - n_i n_m)$$

we can differentiate in (12) to obtain

$$\begin{aligned} \Lambda_{mik} = {} & \frac{\theta \sin \theta}{L} (n_m n_i n_k - n_m\, \delta_{ik}) \\ & + \frac{1 - \cos \theta}{L} (n_k\, \delta_{im} + n_i\, \delta_{km} - 2 n_i n_k n_m) \\ & - \frac{\epsilon_{ikl}}{L} [\sin \theta\, \delta_{lm} + (\theta \cos \theta - \sin \theta)\, n_m n_l] \end{aligned} \tag{14}$$

Equations (10), (13), and (14) allow us to compute the indeterminacy at time t in $\mathbf{L}_1$ and $\mathbf{L}_2$ arising from the original "quantal" indeterminacies ΔL^0_{2i}. As an example, we consider the following initial conditions:

$$\begin{gathered} \mathbf{L}_1{}^0 = (0, L/2, L/2), \qquad \mathbf{L}_2{}^0 = (L/\sqrt{2}, 0, 0) \\ \Delta L^0_{2z} = -\Delta L^0_{2y} = \Delta; \qquad \Delta L^0_{2x} = 0 \\ n_z = n_y = 1/2; \qquad n_x = 1/\sqrt{2} \end{gathered} \tag{15}$$

and the angle α in Fig. 3 is 45°.

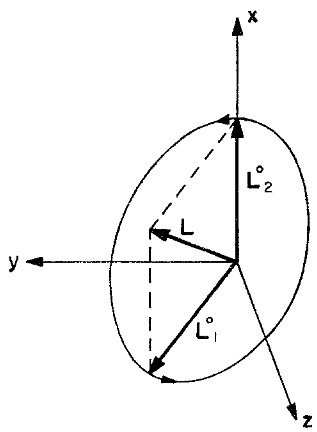

Fig. 4. Initial positions of the $\mathbf{L}_1$, $\mathbf{L}_2$ vectors.

Figure 4 depicts the initial states of 1 and 2.

Now using (13), with the aid of (10), (14), and (15), we obtain the simple results

$$\begin{aligned} \Delta L_{1z}^t &= -\Delta L_{1y}^t = \tfrac{1}{2}(1 - \cos\theta)\,\Delta \\ \Delta L_{2z}^t &= -\Delta L_{2y}^t = \tfrac{1}{2}(1 + \cos\theta)\,\Delta \end{aligned} \tag{16}$$

We can compare the indeterminacy swapping in the quantal case in Fig. 1 with the corresponding phenomenon in this case, as depicted in Fig. 5 for ΔL_{1z}^0 and ΔL_{2z}^0 as functions of t ($t = \theta/kL$).[20]

Now let us write

$$P_1{}^t = \Delta L_{1y}^t \cdot \Delta L_{1z}^t\,, \qquad P_2{}^t = \Delta L_{2y}^t \cdot \Delta L_{2z}^t$$

Consistently with the minimal quantal indeterminacy relation for angular momentum operators in QM, we have for the "quantal" agent 2 at $t = 0$

$$|\,P_2{}^0\,| = \Delta^2 = \tfrac{1}{2}\hbar\,|\langle L_{2x}^0\rangle| \tag{17}$$

[20] We may note in passing that the clean indeterminacy swapping of the two spins in our example is due to the fact that the initial uncertainties in the components of $\mathbf{L}_2$ are chosen in such a way as to not affect $|\,\mathbf{L}\,|$, which controls the speed of precession. In general this would not be the case. We have chosen the simplest example that reproduces the main features of the quantum case: time dependence of $\langle L_{1x}\rangle$, with an initial value of zero, and clean indeterminacy swapping.

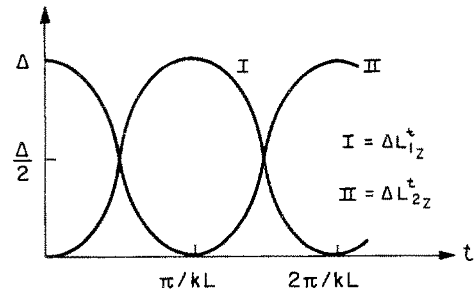

Fig. 5. Spin indeterminacy of systems 1, 2 plotted against time.

From the initial conditions (15) we have

$$L_{2x}^0 = L/\sqrt{2}$$

which yields from (17) the relation

$$L = 2\sqrt{2}\,\Delta^2/\hbar \tag{18}$$

Now, if the DTI holds for time t, then we expect

$$|P_1{}^t| \geqslant \tfrac{1}{2}\hbar\,|\langle L_{1x}^t\rangle| \tag{19}$$

Moreover, using (9), we obtain

$$\langle L_{1x}^t\rangle = (L/2\sqrt{2})(1-\cos\theta) \tag{20}$$

So from (18)–(20) we conclude that the DTI obtains if

$$|P_1{}^t| \geqslant \tfrac{1}{2}(1-\cos\theta)\,\Delta^2$$

However, the solutions (16) in fact yield

$$|P_1{}^t| = \tfrac{1}{4}(1-\cos\theta)^2\,\Delta^2$$

It follows that the object system 1 gains indeterminacies by way of interaction with 2 which are consistent with the quantal indeterminacy relations only for interaction periods corresponding to values of θ that are integral multiples of π (or $t = n\pi/kL$, $n = 1, 2, 3,\ldots$). In Fig. 6 we plot the curve (I) corresponding to the variation with time of the actual value of $|P_1{}^t|$ and the curve (II) corresponding to that of the required value of $|P_1{}^t|$ to be consistent with the DTI. Figure 6 may be compared with Fig. 2 in the quantum mechanical treatment.

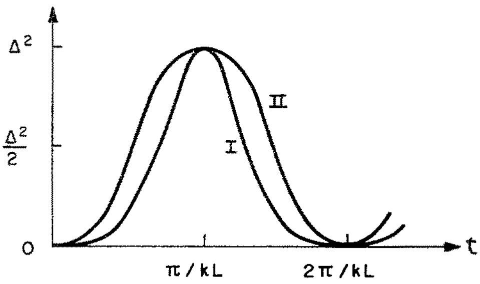

Fig. 6. The actual (I) and required (II) values of $|P_1{}^t| = |\Delta L_{1y}^t \cdot \Delta L_{1z}^t|$ plotted against time.

In contrast to the linear collision example, consistency with the DTI is not retained once it is has been achieved, but is lost and regained in a cyclical manner. This difference in behavior follows from the different geometries of the two interaction processes: the open-ended linear collision and the periodic precession of the two spins about their vector sum.

4. FURTHER CONSIDERATIONS

In Section 3 we were concerned with technical considerations in connection with the explicit "disturbance" process in the DTI as it is expressed in the quasiclassical treatment of interaction processes. On a more general level, one may question the theoretical coherence of the notion in the DTI that a quantal agent is responsible in measurement or state preparation for indeterminacy in the object system. Specifically, it may be asked from where the quantal indeterminacy inherent in the state-preparer originates. The DTI would appear to suggest another state preparation procedure, in which the object system is the quantal agent in the original state preparation process. But this introduces an endless regress. This regress is not avoided in the case, say, of Heisenberg's microscope experiment by arguing that quantal indeterminacy is the intrinsic birthright of the photon, which is then transmitted to matter in the given interaction. It was seen in Section 1 that one can equally regard in the context of the DTI the quantal features of the electromagnetic field as acquired from quantal fluctuations inherent in the material test particle.

If we have succeeded in all the foregoing to place in doubt the tenability of the DTI, it is worth mentioning a wider consequence of this result. Let us consider the question of the necessity of quantizing the radiation field. It was suggested by Henley and Thirring[28] that the Bohr–Rosenfeld result (see Section 1) effectively proves that inconsistencies arise unless the electromagnetic field is quantized. This suggestion provoked a strongly worded denial by Rosenfeld,[29] who stressed that the necessity of quantization is based ultimately on empirical and not logical considerations. Rosenfeld argued that the Bohr–Rosenfeld result established obly "the consistency of the way in which the mathematical formalism of a theory embodying... quantization is linked with the classical concepts on which its use in analyzing the phenomena rests." A position similar to that of Henley and Thirring was taken also by Heitler[30] and more recently by Kalckar.[31]

Although not all the arguments in favor of the necessity of field quantization advanced by these authors on the basis of the Bohr–Rosenfeld result are strictly identical, the main element of reasoning seems to be the following. If, in accordance with the disturbance theory, fluctuations in the field components are brought about by a perturbation caused by the quantal test particle, then in the formal field theory the field components must obey the appropriate commutation relations, in precisely the way that the Heisenberg *gedanken* experiments demand that q-numbers, rather than c-numbers, represent the dynamical variables associated with material bodies. According to this viewpoint, the ineradicable and uncontrollable fluctuations which prevent a measurement from being effectively disturbance-free are the hallmark of the quantal properties of the domain of reality under consideration. It can be concluded that this argument for field quantization, resting as it does on the DTI, is as weak as the disturbance doctrine itself.[21]

The source of the indeterminacy relations is not to be sought then in the disturbance by interaction with another quantal system, but rather is an inherent property of the first system along the lines suggested by Heisenberg's

[21] We are maintaining then that there is no inconsistency in allowing interaction between a quantal and a nonquantal system (or indeed between systems with different values of $\hbar$; cf. Messiah,[14] p. 149, who argues that this would lead to inconsistency). However, the detailed theory of such an interaction would, of course, be subject to certain constraints. Taking again the case of the single-slit experiment with a nonquantal slit, we could not correlate a single position and a range of momentum values of the slit simultaneously with corresponding position and momentum eigenstates of the electron—indeed the momentum variable of the slit might now have to be equated with the expectation value for the momentum of the electron. Thus conservation of momentum would be satisfied only on the average, as between slit and electron. So the detailed formalism of such a hypothetical hybrid theory might involve some unusual physical effects, but would not lead to logical incoherence. Eppley and Hannah[32] argue that the interaction of a quantal and a nonquantal system implies giving up one or another of several alternative principles, such as the strict conservation of momentum we have just described, but they include among their alternatives the uncertainty relations for the quantal system. This possibility is in our opinion ruled out, since, by hypothesis, the quantal system has no states violating the uncertainty relations to get itself into.

original reasoning about the single-slit experiment. Instead of modifying the argument in this case to correspond to the situation in the other *gedanken* experiments as suggested by Bohr, we would recommend exactly the reverse emendation.

REFERENCES

1. W. Heisenberg, *The Physical Principles of the Quantum Theory* (Dover, New York, 1949).
2. M. Jammer, *The Philosophy of Quantum Mechanics: The Interpretations of Quantum Mechanics in Historical Perspective* (John Wiley, New York, 1974).
3. C. Roychoudhuri, *Found. Phys.* **8**, 845 (1978).
4. N. Bohr, Discussion with Einstein on epistemological problems in atomic physics, in *Albert Einstein: Philosopher Scientist*, P. A. Schilpp, ed. (Tudor Publishing Co, New York, 1949), p. 199. [Reprinted in N. Bohr, *Atomic Physics and Human Knowledge* (Wiley, New York, 1958), p. 32.]
5. N. Bohr, *Phys. Rev.* **48**, 696 (1935).
6. E. Scheibe, *The Logical Analysis of Quantum Mechanics* (Pergamon, Oxford, 1973).
7. N. Bohr and L. Rosenfeld, *Kgl. Dan. Vid. Sel. Mat. Fys. Medd.* **12**, No. 8 (1933).
8. H. Robertson, *Phys. Rev.* **29**, 163 (1929).
9. E. Schrödinger, *Berliner Berichte* **1930**, 296.
10. J. Synge, *Proc. Roy. Soc. (Lond.) A* **325**, 151 (1971).
11. H. Margenau, *Phil. Sci.* **30**, 1, 138 (1963).
12. K. Popper, in *Quantum Theory and Reality*, M. Bunge, ed. (Springer, Berlin, 1967), p. 7.
13. L. Ballentine, *Rev. Mod. Phys.* **42**, 358 (1970).
14. A. Messiah, *Quantum Mechanics*, Vol. 1 (North-Holland, Amsterdam, 1961).
15. J. Bub, in *Logic and Probability in Quantum Mechanics*, P. Suppes, ed. (Reidel, Dordrecht, 1976), p. 397.
16. J. Park, *Found. Phys.* **1**, 23 (1970).
17. A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
18. D. Bohm and Y. Aharonov, *Phys. Rev.* **108**, 1070 (1957).
19. M. Renninger, *Z. Phys.* **158**, 417 (1960).
20. Y. Yoshihuku, *Mem. Chubu. Inst. Technol. (Japan)* **13-A**, 173 (1977).
21. H. Reichenbach, *Philosophical Foundations of Quantum Mechanics* (University of California Press, Berkeley, 1944).
22. P. Feyerabend, *Philosophical Studies* **9**, 47 (1958).
23. C. Hooker, *Australasian J. Phil.* **49**, 262 (1971).
24. D. Bohm, *Quantum Theory* (Prentice Hall, New York, 1951).
25. P. Fong, *Elementary Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1962).
26. M. Gardner, *Phil. Sci.* **38**, 508 (1971).
27. H. Jeffreys and B. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, London, 1962).
28. E. Henley and W. Thirring, *Elementary Quantum Field Theory*, (McGraw–Hill, New York, 1962).
29. L. Rosenfeld, *Nucl. Phys.* **40**, 353 (1963).
30. W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1954).

31. J. Kalckar, in *Foundations of Quantum Mechanics* (International School of Physics "Enrico Fermi," Course 49), B. d'Espagnat, ed. (Academic Press, New York, 1971), p. 127.
32. K. Eppley and E. Hannah, *Found. Phys.* **7**, 51 (1977).
33. F. Herbut, *Int. J. Theor. Phys.* **11**, 193 (1974).