3

"Philosophical" Intermezzo I: What Is Determinism?

3.1 Definitions

Since we used the word "determinism" in the previous chapters, it might be good to pause for a moment and to define what this word means and what it implies.

3.1.1 Determinism and Randomness

To define determinism in a physical theory, one can simply say that, if the state of a physical system is given at some time, then there are laws specifying what this state will be at all later times.

When we say that "there are laws", we do not assume that we know those laws but simply that certain systems follow certain laws. After all, we should have no doubt that the planets were moving according to the laws of gravitation before we discovered those laws or even before humans existed.

Let us first define the *state* of a physical system. If one considers the sun and one planet (forgetting about everything else in the Universe), and if the precise positions and velocities of both bodies are given, then their positions and velocities are determined for all future times by Newton's laws of motion, and they can even be expressed in a rather explicit formula.[1] If one considers the sun and all the planets (again, forgetting about everything else) and one specifies the precise positions and velocities of all these bodies at a given time,

[1]The reason why one need to specify both positions *and* velocities and not positions alone is due to a property of Newton's laws, that we will not go into.

J. Bricmont, *Quantum Sense and Nonsense*, DOI 10.1007/978-3-319-65271-9_3

then their positions and velocities are again determined for all future times by those laws of motion, but there does not exist an explicit formula expressing what those positions and velocities will be.

This extends in principle to the whole Universe, as long as one considers only gravitational forces and one remains within the framework of classical physics.

This also extends to the tossing of a coin, the throwing of a dice or a roulette ball on the wheel: if one specifies exactly the way those objects are thrown (their position, velocity, the way they rotate etc.), then again their future is determined.

In these examples, the state of the systems under consideration is defined by the exact positions and velocities of all the bodies of the system at a given time. Once such a state is specified, the laws of motion determine the state of the system at all later times.

The important word here is "exact": everyone knows that, if the coin, the dice or the roulette ball are thrown a little bit differently, the result may be heads instead of tails, a 6 instead of a 2, or the ball landing on the red instead of on the black.

The essential point that defines a deterministic dynamics is that it associates to each exact state of the physical system at a given "initial" time a *unique* state for all future times. We will call *initial conditions* the exact state of the system at that initial time.

In physics, one also considers non-deterministic dynamics. This is defined as a dynamics where each initial state does *not determine* a unique later state, but several possible states.

It is a dynamics defined by assigning to each state a series of other states with certain probabilities.

The simplest examples are again given by coin tossing and dice throwing. Suppose that we do not know (as is the case in practice) the initial conditions in a particular tossing or throwing. Then, we assign certain probabilities to the outcomes.

In the case of a coin, assuming that there are no tricks or bias, the probability of heads or tails is $\frac{1}{2}$, irrespective of what happened previously. For a dice, each face has probability $\frac{1}{6}$, again irrespective of what happened previously.

But one could imagine a biased coin, where heads would fall with probability $1/4$ and tails with probability $3/4$. Or a biased dice where one face would appear with probability $1/2$ and each of the five other faces with probability $1/10$. Obviously, the sum of the probabilities of the different possible outcomes must always be equal to 1.

Almost all of the applied sciences use probabilistic dynamics, of course more fancy ones than the examples given here, but based on the same idea.

The outcomes of such non-deterministic processes are often called "random". To define (intuitively) a random sequence of results, consider the repeated tossing of a non biased coin, with results noted H (heads) and T (tails). This sort of experiment is typical of what one calls random and thus can be used to explain that notion.

What does one expect when one tosses many times such a coin? First of all, one expects both H and T to appear about half of the time.[2] But one expects also each successive pair of the form HH, HT, TH, TT to appear about a quarter of the time. Each of the eight series composed of three successive results HHH, HHT, HTH, etc. should appear one-eight of the time. More generally, in a random sequence, every finite series of symbols, made of H's and T's, should appear with a frequency that depends only on its length (the longer the length, the smaller the frequency)[3] so that two finite series of the same length appear with the same frequencies.[4]

This notion of random sequence may become more intuitive if one considers examples of sequences that are not random. For example, the sequence HTHTHTHT…is just the repetition of the pair HT, which therefore occurs more than a quarter of the time, and is not random. Another example is given by the sequence HHTTHTHHHTTHTHHHTTHTHHHTTHTHHHT-THTH…, which is obtained by the repetition of the series HHTTHTH, and is also not random (the series HHTTHTH is of length 7 and thus should occur, in a random sequence, only a fraction $\frac{1}{2^7} = \frac{1}{128}$ of the time).

We will call *apparently random* (for reasons that will be clear below) a sequence satisfying the above definition.

Suppose we are given a sequence that is apparently random according to the above definition. We can raise a fundamental question. Is that sequence *truly random*, or, to use a synonym, *intrinsically random*? What these expressions mean is that no conceivable deterministic explanation of the appearance of this random sequence could be given. This notion has to be contrasted with the previous definition (apparently random) that referred to a statistical property of the sequence (two finite series of the same length appear with the same

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[2]This expectation will be justified trough the law of large numbers in Sect. 3.4.1.

[3]To define the frequency with which some series of symbols occurs in a sequence of results, one counts the number of those series of symbols in that sequence and one then divides that number by the total number of results.

[4]Thus, every finite series of results of length n will occur a fraction of the time equal to $\frac{1}{2^n}$. This of course makes sense only for an infinite random sequence. Since, in practice, every sequence has a finite length, the definition given here has to be considered as an idealization.

frequencies), while the notion of truly or intrinsically random refers to the impossibility for the sequence to be produced by a deterministic mechanism.

We already saw that the results of coin tossing are not *truly* random. This gives a simple example of a sequence of results that appears random, but that is in reality deterministic: if we give a more detailed description of the system, namely the exact initial conditions of the state of the coin for each tossing, then the results are determined by those initial conditions.

Conversely, any deterministic system can look indeterministic if we do not describe it in sufficient detail. The coin tossing and the dice throwing described above provide examples of that situation. If we give the initial conditions in detail, then those processes are deterministic, but if we do not, then they may *appear to be* random.

One might ask a further question: can one find a criterion that would allow us to determine, whenever the behavior of a system is *apparently random*, whether it is *truly random*? The answer is "no": although we shall not prove it, the examples of the coin tossing and the dice throwing illustrate why this is so: one cannot think of anything more apparently random than the throwing of a coin or a dice. In fact these examples epitomize the notion of randomness. But if even those system are in fact deterministic, once one gives a more complete description of their states, how can one hope to find a criterion that would prove that a system is truly random and does not simply look random because of our incomplete description of their states?

This does not mean that there cannot be any truly random systems in Nature, but that, in order to prove that fact, one has to give other arguments than simply observe that they appear random.

3.1.2 Determinism and Predictability

This brings us to another important distinction, the one between *determinism* and *predictability*. In the examples of the coin tossing or the dice throwing, we said that, if we knew the initial conditions with enough precision, we could in principle calculate on which face they would fall. But here and in many other phenomena, we cannot, in practice, know the initial conditions with enough precision to be able to predict the face on which they will fall.

It may also happen that, although the initial conditions can be known with great precision, the calculation of the evolution of the system between its initial state and its final one may be too complicated to be carried out.

Yet another possibility is that *there are deterministic laws* governing a given physical situation, but that we do not know them.

To summarize the difference between determinism and predictability, the first notion refers to the nature of the (possibly unknown) laws governing a given phenomenon and the second one to our ability to know and use those laws to predict the future. So, predictability does not depend only on the nature of the physical laws (whether they are deterministic or not), but also on human abilities, both our capacity to make precise measurements (of initial conditions) and to make possibly complicated calculations.

The French mathematician Henri Poincaré explained nicely the difference between predictability and determinism in the following passage:

> Why have the meteorologists such difficulty in predicting the weather with any certainty? Why do the rains, the tempests themselves seem to us to come by chance, so that many persons find it quite natural to pray for rain or shine, when they would think it ridiculous to pray for an eclipse? We see that great perturbations generally happen in regions where the atmosphere is in unstable equilibrium. The meteorologists are aware that this equilibrium is unstable, that a cyclone is arising somewhere; but where they can not tell; one-tenth of a degree more or less at any point, and the cyclone bursts here and not there, and spreads its ravages over countries it would have spared. This we could have foreseen if we had known that tenth of a degree, but the observations were neither sufficiently close nor sufficiently precise, and for this reason all seems due to the agency of chance. Here again we find the same contrast between a very slight cause, unappreciable to the observer, and important effects, which are sometimes tremendous disasters.
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> Henri Poincaré [153, p. 398] (original [152])

Events that look random or chancy to us, like the weather, may very well be as determined as the eclipses and this appearance of randomness then reflects only our cognitive limitations.

Because of what is said above, some readers may draw the conclusion that physical laws *are necessarily deterministic* if one describes the physical situation in sufficient detail or, in other words, that the notion of having fundamental physical laws that are indeterministic is unthinkable. But we made no such claim.

Indeed, although it is difficult to give a criterion showing that some systems behave in a "truly random" fashion we should not assume that a fundamentally "random" Universe is impossible. It is perfectly conceivable that the fundamental laws of Nature are not deterministic. Who are we to tell Nature how to behave?

People who do not like determinism sometimes accuse it of being a "metaphysical assumption". Since we don't assume determinism or demand a priori that a physical theory be deterministic, this accusation cannot be used against the point of view defended here.

3.2 Determinism and Physics

Of course, the coin tosses and dice throwings discussed above are too simple to be of any interest in physics.

When Newton and his successors discovered what is known as classical mechanics (basically the laws of motions of planets, projectiles, satellites) they introduced the archetypical example of *deterministic physical laws*.

This led the French physicist and mathematician Pierre Simon Laplace to write, at the end of the 18th century:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.[5]
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> Pierre Simon Laplace, [116, p. 4]

This is the clearest formulation of the universal determinism of the laws of classical physics. But, to come back to the distinction that we made between determinism and predictability, note that Laplace added immediately after that sentence that *we* shall "always remain infinitely removed" from this imaginary "intelligence" and its ideal knowledge of the "respective situation of the beings who compose" the natural world; that is, in modern language, ideal knowledge of the exact initial conditions of all the particles in the Universe.

Laplace distinguished clearly between what Nature does and our knowledge of it. Moreover, he stated this principle at the beginning of an essay on *probability*. But, as we shall discuss below, probability for Laplace is nothing but a method that allows us to reason in situations of partial ignorance.

The meaning of Laplace's quote is completely misrepresented if one imagines that *he* hoped that one could arrive someday at a perfect knowledge and a universal predictability, for the aim of his essay was precisely to explain how to proceed in the absence of such a perfect knowledge.

[5]This intelligence is often referred to as the "Laplacian demon". (Note by J.B.).

However, celestial mechanics dealt only with the motion of bodies attracting each other through gravitational forces.

In the 19th century, physicists discovered laws governing electric and magnetic phenomena. As we said in Sect. 1.1.1, James Clerk Maxwell, expressed those laws through a mathematical theory, known as electromagnetism, postulating the existence of electromagnetic waves, that are both created by the particles and that guide their motion.

The system consisting of particles and waves interacting with each other is still deterministic in the same sense that classical mechanics is. The only novelty is that one has to specify not only the initial positions and velocities of all the particles in the system, but also the initial state of the electromagnetic waves.

Physics was further modified in the beginning of the 20th century by the two theories of relativity, the special and the general ones, due mainly to the works of Lorentz, Poincaré, Einstein and Hilbert.

These theories basically changed Newton's laws of motion (in order to make them compatible with the newly discovered laws of electromagnetism) and changed also Newton's theory of gravitation.

But as far as determinism is concerned, nothing substantial changed.[6]

The real break with determinism in physics came with quantum mechanics. In the previous chapter, we saw that we cannot predict the exact place where the particle is going to end up on the second screen in the double-slit experiment.

In the next chapter, we shall see that the formalism of quantum mechanics incorporates this unpredictability in its very formulation: one associates to physical systems certain states and, given such a state, one can compute the probabilities of jumping into another state after a "measurement". But, at least in ordinary quantum mechanics, there is no information that would, *even in principle*, allow us to predict with certainty which state one will jump into. Thus, it seems that, with quantum mechanics, we have a candidate for a physics governed by intrinsically random laws.

Of course, and now we come back to what we said about the coin and the dice: could it be that, by providing a more complete description of the physical systems than the one of ordinary quantum mechanics, one could recover a deterministic theory?

So, the issue of determinism and of whether a more detailed description than the usual quantum one is possible, are intimately linked.

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[6] There are several caveats that should be made here, in order to be rigorous, but it would go far beyond the scope of this book.

And these two questions were at the heart of the discussions between Einstein, Schrödinger, Bohr, Heisenberg, Pauli and others.

But we shall return to these issues later and discuss now the reason why determinism seems to many people to have great negative "philosophical" consequences.

3.3 Determinism and Free Will

The notion of universal determinism of physical laws has provoked much hostility because it seems to contradict our notion of free will. To discuss that issue, we must first try to define free will: free will is not just the feeling that I act without external constraints, without anybody forcing me to do or not do something, but that "I" choose to do one thing rather than another.

The choice can be trivial, like between two kinds of deserts, or serious, like which profession to embrace or whether or not to commit a crime. But in our everyday life, we constantly feel that we have to make choices and, when we are not under external constraint, that we do so freely.

This notion of free choices has very many moral and legal implications. Courts routinely distinguish between people who are responsible for a crime that they committed and those who are not. The first ones are sent to jail, the others to mental hospitals.

But suppose that someone tells you that those who commit crimes apparently "of their own free will" are actually determined to commit those acts and are not "really" free not to do them.

How could that be possible? Let us first discuss a form of "determinism" which is very common in social and political discussions, but which is not relevant here: social and psychological determinism. For example, one might say that some person had a bad childhood or grew up in a crime-prone environment and thus became a criminal. The problem with this form of "determinism" is that it is not strict: one can always find counterexamples, namely people who have had the same psychological or sociological background and that did not become criminals. Even if one adds what one knows about genetic determinism to the picture, the result is still not strict determinism and, presumably, even a more detailed knowledge of genetics than the one we have now would not lead to a strict determinism.

In fact, these forms of "determinism", or rather of lack of strict determinism, are again examples where this lack is due to an incomplete description of the systems.

To give a more detailed description, for which determinism might hold, one should describe the state of every brain cell and every connection between cells in every detail, maybe down to the state of the last atom of our brain, and also describe all the interactions between the brain, the rest of the body and the outside world. Such detailed states would almost certainly differ between people of the same socio-psychological background, and that would lead one of them to commit a murder and the other not.

But then, of course, one may say that the person who commits the murder is not more free not to do it than the person who does not commit it: in both cases, we have zillions of neurons interacting with each other and producing different results, but that is just like two computers executing different programs. There is no genuine free will in either situation.

Of course, such a detailed description would in practice be impossible to obtain, but remember that we discuss determinism here, not predictability.

This leads us to the view sometimes known as the "clockwork universe": at the time of the Big Bang, the world started in a certain state and then it evolved deterministically according to physical laws.

This view has all kinds of strange implications: while I try to find the next words to write in this book, they were in fact all determined at the time of the Big Bang.

Moreover, our strong feeling that some criminals are really guilty, because they acted out of their own free will, while others may simply be mentally disturbed, would become an illusion if what is said above were true. It would only be that our knowledge of brain states is insufficient to explain what goes wrong in the brains of the people considered "really guilty". Note that, in the same way that there is no test that can show that some events are "truly random", nobody has ever given a way to test whether some individual's actions are really due to his or her "free will".

But if one accepts what precedes, and although we may deplore the acts of criminals, it would be natural to do it in the same way that we deplore the effects of hurricanes. If the Universe is deterministic, there is no reason to hold the worst criminals responsible for their actions; after all, we do not hold hurricanes responsible for their effects. If there is no genuine free will, what is the difference? While we are at it, why hold Hitler responsible for his actions?

In the same spirit, one might also wonder why saints and heroes deserve to be admired ? After all, the difference between them and the worst criminals is simply that they have bunches of neurons in different states. One could admire them in the same way that one admires a computer playing chess very well, but that is usually not the same admiration as the one extended to humans.

One could reply that this does not make any difference in practice: nobody could even dream of ever predicting what an individual is going to do (in general circumstances) by analyzing its brain and "computing" its future behavior.

Besides, our legal judgments, our sentiments of love or hatred, of admiration or contempt, are just as determined by Nature as anything else; so there is nothing that we can change about them.[7]

But this does not remove the uneasiness created by the conflict between determinism and free will.

And this is where quantum mechanics comes in: if quantum mechanics shows that Nature is truly random and if, when we come down to the microscopic structure of anything, including our brains, we encounter atoms or electrons, and if the latter obey the laws of a "truly random" quantum mechanics, then, says the defender of free will, the argument going from universal determinism to "free will is an illusion" collapses.[8]

This is correct, but a universe whose ultimate laws are truly random will not give us a picture of the world in which free will exists either. Our feeling of free will does not mean that there are some random events in our brains, but that *conscious choices* are made.

Those have nothing to do with random events. The problem of free will is part of what philosophers call "the mind-body problem": we see ourselves as having feelings and sensations and making free choices, while our view of the physical universe has no such properties: the world outside ourselves looks like a machine and whether it has some random element in its functioning does not change anything so far as that is concerned.[9]

One "solution" to that problem is dualism: the body and the mind belong to two different substances (in the Western religions, the mind is identified with the soul, which is supposed to be immortal, but one can be a dualist without believing in the immortality of the soul/mind). But that solution creates problems of its own: does the mind interact with the body? If it interacts, does it violate physical laws in doing it? If it does not interact, what is the point of introducing the mind as a substance separate from the body?

Besides, if this is supposed to explain free will, how and when did the latter appear during evolution? One can have longer or shorter arms or necks, but it does not seem to make sense to have half of genuine free will. Either you have

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[7] Nevertheless, one can argue that the development of a scientific world view has led to an increased skepticism with respect to the doctrine of free will and that has led to changing attitudes with respect to education and to criminal law: "rewards and punishments" have been increasingly viewed as a practical matter rather than a question of principle based of distinguishing true merit and true guilt.

[8] For example, the distinguished physicist Nicolas Gisin links the supposed lack of determinism in quantum mechanics to free will, see [90, 119].

[9] We will come back to the mind-body problem in Sect. 11.5.1.

it or you don't. Were some animals devoid of free will, while their children had it? Or is every living being endowed with free will? What about plants?

The point of this digression is only to show the depth of the problem, which we have no intention of solving or even of discussing any further.[10]

Indeed, our only goal here is to "free" physics, so to speak, from concerns about free will. We do not want to be distracted by the frequent feeling that an indeterministic physical theory is preferable to a deterministic one, because it would allow us to "save free will". There is nothing in the character of physical theories that will make free will more or less understandable or plausible. So let us pursue our investigation of physical theories and let the chips fall where they may, without philosophical prejudices or worries.

Besides, as the British logician and philosopher Bertrand Russell observed, scientists should look for deterministic laws like mushroom seekers should look for mushrooms. Deterministic laws are preferable to non-deterministic ones because they give a way to control things more efficiently (at least in principle) and because they give more satisfactory explanations of why things are the way they are. Looking for deterministic laws behind the apparent disorder of things is at the heart of the scientific enterprise.

Declaring some events truly "random" is basically saying that we have no explanation for them and that is not what science hopes to do (although, of course, one may be forced to concede that we fail to find an explanation for certain events).

Whether we succeed or not in that search for deterministic laws depends in a complicated way both on the structure of the world and on the structure of our minds. But the opposition to determinism tends to make people feel that the project itself is doomed to fail or even hope that it will fail. And that state of mind does run counter to the scientific spirit.

3.4 Probabilities and Determinism

Now, let us go back to the idea of deterministic laws and ask ourselves, why and how do physicists use probabilities, as they do in practice, if laws are deterministic?

The answer was already given when we mentioned the limitations of our abilities: probabilities are used because of our ignorance: either because we do not know the initial conditions of the system or because, even if we knew them, the calculation of the future would be too complicated.

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[10]The philosopher Colin McGinn has developed the interesting idea that the problem of "free will" may lie beyond the limits of human understanding [123].

But how does one assign probabilities to an event? If we consider simple examples, like coin tossing or dice throwing, we can use the symmetry of the situation: there are two faces for the coin and six for the dice and we do not see any reason why one would be more likely than the other, so we assign probability one-half for each face of the coin and one-sixth for each face of the dice.[11]

As we said, in any single throw of the coin or the dice, the result is determined by the initial conditions and would be predictable if one could know them with sufficient precision. The fact that we assign probabilities to such events does not mean that we deal with something "truly random", but simply that the phenomena in question are not controllable or computable in detail by us.

In more complicated situations, where there is no symmetry between the different outcomes, assignment of probabilities is more difficult and we shall not explain in detail how one does it. What one tries to do is to apply Laplace's "principle of indifference", namely one tries to reduce the situation to a set of cases about which we are "equally ignorant", i.e., the information that we do have does not allow us to favor one case over the other. In other words, we try to get as close as possible to the situation of the tossing of a coin or the throwing of a dice, where there is a symmetry between the faces.

What one wants to avoid, when assigning probabilities, is to introduce bias in our judgments, or "information" that we do not really have or is illusory, like people who believe in lucky numbers.

In physics, if one deals with a deterministic dynamics, but where the initial conditions are in practice unknown, one assigns a probability distribution to those initial conditions and that determines a probability distribution on future events: the probability of those events is simply the probability of the initial conditions that led to them through the deterministic laws.[12]

But simply quantifying our ignorance, which is what assignments of probabilities do, may not sound very useful. In order to obtain useful probabilistic statements, we need the law of large numbers.

3.4.1 The Law of Large Numbers*

That law says, roughly speaking, that, even if a sequence of events are random, there is some regularity that emerges when the same random events occur many times.

[11] In case our observations systematically deviate from what one would expect on the basis of this assignment of probabilities, namely that all faces fall with equal frequencies, we will suspect the coin or the dice to be biased and then revise our probabilities.

[12] This notion will be important in Sect. 8.4.2 when we discuss the de Broglie–Bohm theory.

To explain that law through a simple example, consider coin tossing: if one tosses a coin many times, say n times, one gets a sequence of results, for example: H, T, T, T, H, H, ..., (with H = heads, T = tails). Each such sequence has the same probability, namely $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \cdots = (\frac{1}{2})^n$, since each individual result has probability $\frac{1}{2}$ and there are n tosses of the coin.

For example, if $n = 4$, we have $(\frac{1}{2})^4 = \frac{1}{16}$; if $n = 10$, we have $(\frac{1}{2})^{10} = \frac{1}{1024}$.

Now, let us compute the probability that the coin always falls heads. That has also probability $(\frac{1}{2})^n$, since there are n tosses of the coin and, in each case, falling heads has probability $\frac{1}{2}$. The same holds for the coin always falling tails.

On the other hand, let us compute the probability of the coin falling heads half of the time (and thus tails also half of the time), assuming n to be an even number? Unlike the situation where the coin always falls heads or tails, here there are many different sequences of outcomes where the number of heads is $\frac{n}{2}$. For example, for $n = 4$, one can have: H, T, T, H, or H, T, H, T and four other possibilities.

There is a mathematical formula that counts the number of sequences of outcomes where the coin falls heads half of the time.[13] There is also a formula that counts the number of sequences of outcomes where the coin falls heads k times and tails $n - k$ times.[14]

Now, using those formulas one can show that the number of sequences of outcomes where the coin falls heads (and tails) more or less half of the time is, for n large, almost equal to the total number of possible sequences.[15]

That may sound strange at first sight, but it is just the result of a computation: for n large, the overwhelming majority of sequences of results have a number of heads more or less equal to the number of tails and both are more or less equal to $\frac{n}{2}$.

But that means that the probability that a given sequence of results has a number of heads more or less equal to the number of tails is almost equal to one, or, in other words, is almost certain.

This result is the simplest example of the law of large numbers (which we will not state in general, but only through examples).

[13]We shall not use that formula, but we give it for the interested reader: writing $n = 2m$, since n is even, that number of sequences is: $\frac{n \times (n-1) \times (n-2) \cdots \times (m+1)}{1 \times 2 \times 3 \cdots \times m} = \frac{n!}{m!^2}$, where, for a number m, m! is a shorthand notation for $1 \times 2 \times 3 \cdots \times m$.

[14]Again, for the interested reader, that number is: $\frac{n!}{k!(n-k)!}$.

[15]Which is equal to $2 \times 2 \times 2 \cdots = 2^n$. We shall not be precise about what "large", "more or less" and "almost equal" mean here. For a mathematician, n "large" means that one considers a limit where n tends to infinity, "more or less" means that the frequencies of outcomes where the coin falls heads is very close to $\frac{1}{2}$, and "almost equal" means that the equality becomes exact in the limit where n tends to infinity.

Another example of the law of large numbers is that, in the overwhelming majority of sequences of results, each of the series HH, HT, TH, TT occurs more or less one quarter of the time. And each of the eight series HHH, HHT, HTH, etc. occurs, in the overwhelming majority of sequences of results, one-eight of the time.

Let us define the *statistical distribution* of a sequence of results obtained by repeating a large number of times the same experiment: one collects all the frequencies with which different events occur in a given sequence of results.

For coin tossing, events would be a H or a T, a series made of a pair HH, HT, TH,TT, or a series made of a triple HHH, HHT, HTH, etc. So, one counts the number of H and T, the number of pairs HH, HT, TH,TT, of triples HHH, HHT, HTH, etc. and one divides that number by the total number of results. We call that set of data (all the frequencies of occurrence of finite series of H and T) the statistical distribution of the given sequence of results.[16]

The general law of large numbers says that, when one repeats a large number of times the same experiment, associated to a given probability distribution (here, for coin tossing, each face has probability one-half), in the overwhelming majority of sequences of results, each event (like a H or a T, or a series HT, HHT etc.) will occur with a frequency equal to its probability: the probability of a single symbol H or T is one-half, the probability of a series of two symbols is $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$, because each tossing is independent of the other and the probability of a series of three symbols is $\frac{1}{8} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, which corresponds to the frequencies mentioned above.

More generally, suppose that there is an experiment whose results are associated to a given probability distribution $P(x)$, as in Fig. 3.1. If one repeats many times that experiment, one will obtain a sequence of results. To define the statistical distribution of those results, divide the x axis into small intervals and count the number of particles in each interval, then divide that number by the total number of particles. This allows us to define a curve, denoted $D(x)$ in Fig. 3.2, such that the area between that curve and an interval A on the x axis gives the fraction of particles in that interval.

There are of course many possible sequences of results, but, in the overwhelming majority of them, the statistical distribution of the results, denoted $D(x)$ in Fig. 3.2, will coincide with the probability distribution, provided of course that this probability distribution has been adequately chosen to reflect the physics of the problem.[17] Of course, since, in practice, the sequences of

[16] Again, this notion becomes precise in the (idealized) limit where the number of results tends to infinity.

[17] In fact, one way to verify that the chosen probability distribution was correctly chosen is through this coincidence between the frequencies of events in most of the sequences of results and that probability.

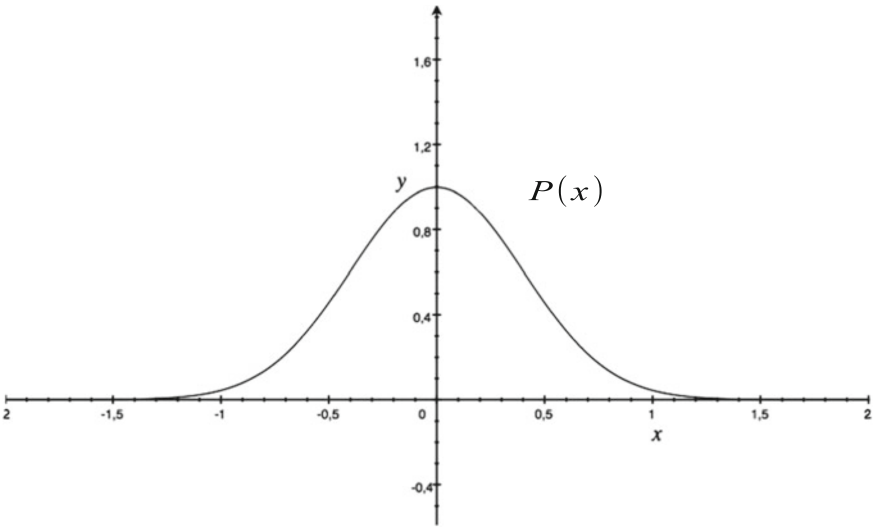

Fig. 3.1 An example of a (Gaussian) probability distribution. See Fig. 3.2 for its relation to statistical distributions

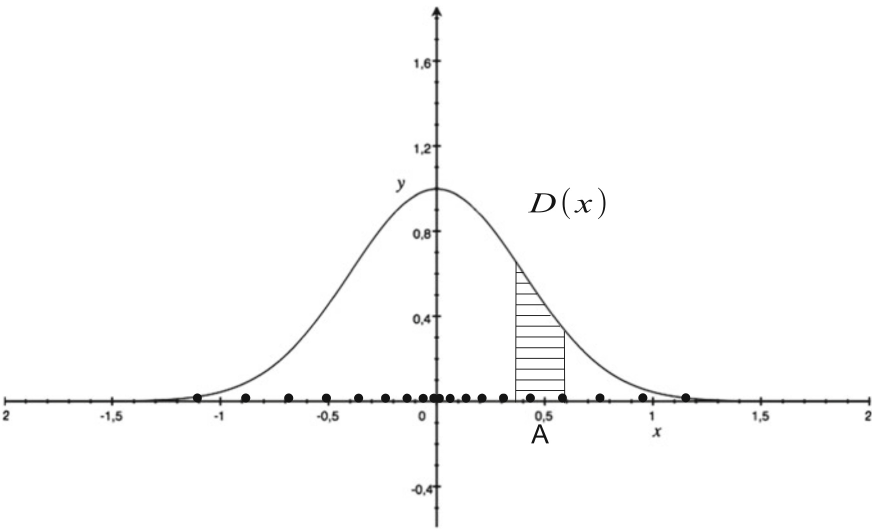

Fig. 3.2 An example of a statistical distribution of a sequence of results of experiments associated to the probability distribution $P(x)$ of Fig. 3.1, each *dot* representing one result. The *curve* $D(x)$ is obtained by dividing the x axis into small intervals and counting the number of particles in each interval, then dividing that number by the total number of particles. The area between the *curve* $D(x)$ and a given interval A on the x axis gives the fraction of particles in that interval

results are always finite, the coincidence between the frequencies and the probability is only approximate, the approximation improving with the number of results.

Coming back to the curves discussed in Chap. 2 and describing the densities of impacts of particles, for example the blue curves in Fig. 2.1 or in Figs. 2.6–2.8, we can also view them as applications of the law of large numbers: each individual particle will land at a certain place, which will be different from one experiment to the next. But, if one sends many particles, one always gets the same curves for the densities of the detected particles, irrespective of whether the experiment is done today in Paris or next week in New York.

This law of large numbers applies to many more complicated situations and is at the basis of most usages of probabilities in physics and in the natural sciences: for example, the whole field of statistical physics, which derives properties of gases, liquids and solids from those of their microscopic constituents (atoms or molecules) is entirely based on the law of large numbers. The properties of gases, liquids and solids look deterministic and are certainly reproducible, but they nevertheless result from the application of the law of large numbers to myriads of atoms or molecules whose motions are not known in detail and are therefore treated as being "random", even though the laws governing their motion may be deterministic and our use of probabilities is only due to our ignorance. The law of large numbers is also used in many applied sciences (insurances for example).

So, even though probabilities are only a way to reason in situations of ignorance, we recover some degree of (almost) certainty when one considers a large number of similar events and one is interested in some "average" behavior. And it is mostly through this law of large numbers that probabilities are useful in physics. We will see how this law can be applied to understand the quantum statistics in Chap. 8.

3.5 Summary

First, we explained that there can be two different sorts of physical laws, deterministic and indeterministic ones. All pre-quantum mechanical laws, those of Newtonian mechanics, of electromagnetism, and of special and general relativity are of the first sort. Only quantum mechanics is a potential candidate for a sort of physical laws where randomness enters at a fundamental level.

(Footnote 17 continued)
For example, if a coin is biased, one will observe a deviation between the observed frequencies and the assigned probabilities, and this should lead us to revise those probabilities.

But we also saw that a process like coin tossing may appear random only because we do not know or do not control the initial conditions with enough precision.

Therefore, the supposedly fundamental randomness of quantum mechanics depends on whether quantum mechanics is "complete" – namely on whether or not a more detailed description of physical systems can be given than the one of ordinary quantum mechanics.

There is a strong opposition to determinism in physics because of our feeling of "free will" and our desire to believe that this free will is genuine and not an illusion. But we argued that even an indeterministic physics would not make free will more genuine. The issue of free will is part of the mind-body problem and lies outside the scope of physics.

Finally, even if the physical laws are deterministic, it does not mean that probabilities do not have to be used. In fact, we have to use them because of our ignorance or because of our computational limitations. When we assign probabilities to events, we try to quantify our ignorance, on the model of a coin tossing or a dice throwing, where there is a symmetry between the faces and we thus assign the same probability to each face.

This may not seem very useful, but the law of large numbers allows us to make statements that are almost certain when one considers a large number of random events, as one does in statistical physics. In that sense, we recover some certainty in the midst of uncertainty.